**Polar Exploration**

Polar coordinates are another way of expressing points in a plane. Instead of being centered at an origin and moving horizontally or vertically, polar coordinates are centered at the pole and measure a radius out from the pole at a given angle. The beginning angle of zero radians corresponds to the positive x-axis.

I. Complete the following formulas based on the diagram. These will be used in the other sections to convert between polar and rectangular forms.

\[
\begin{align*}
\tan \theta &= \\
\cos \theta &= x = \\
\sin \theta &= y = \\
r^2 &= 
\end{align*}
\]

II. Using the equations in part I, convert the following polar coordinates to rectangular coordinates and plot all six points on the two grids provided. The polar grid on the left has the special angles marked. In polar coordinates, the first number tells you how many rings to count out from the center; the second number tells you which radian line to follow.

Polar: \(A \left(2, \frac{\pi}{6}\right)\); rectangular: 
Polar: \(B \left(-2, \frac{7\pi}{6}\right)\); rectangular:

State the polar coordinates of a point C in the same location with \(r > 0; -2\pi \leq \theta \leq 0\):

Polar: \(D \left(3, \frac{7\pi}{6}\right)\); rectangular:
Polar: \(E \left(3, -\frac{5\pi}{6}\right)\); rectangular:

State the polar coordinates of a point F in the same location with \(r < 0; 0 \leq \theta \leq 2\pi\):
Polar Exploration

III. Using the equations in part I, convert the following rectangular coordinates to polar coordinates and plot the points on the grid provided.

Rectangular: $G(0, 3)$; polar: __________
Rectangular: $H(-1, 1)$; polar: __________

How do the locations of polar and rectangular coordinates on the different grid compare?

IV. Convert each rectangular equation to polar and sketch the graph.

$x y = 4$ \hspace{2cm} $x^2 + y^2 - 2x = 0$

V. Convert each polar equation to a rectangular equation and sketch the graph.

$r = 2$ \hspace{2cm} $\theta = \frac{\pi}{4}$
Polar Exploration

VI. Graphing Calculator Experiments:

Experiment #1:

a. \( r = 3, \ 0 \leq \theta \leq 2\pi \); try to answer the following BEFORE you use the graphing calculator, then check your answers by graphing in POLAR mode with zoom decimal or square.

What shape will be graphed?

If you trace, where will the shape start?

If you trace, which direction will you draw?

How many times will the figure be drawn? What do you change to make the figure draw over itself?

b. \( r = a \sin \theta, \ 0 \leq \theta \leq 2\pi \) where \( a = 2, 3, -2 \). Answer the following for each given value of \( a \).

\[
\begin{align*}
r &= _______ \\
\end{align*}
\]

\[
\begin{align*}
r &= _______ \\
\end{align*}
\]

\[
\begin{align*}
r &= _______ \\
\end{align*}
\]

Description: __________

Center: _______

Radius: _______

Starts at _______

Direction: _______

\[
\begin{align*}
r &= _______ \\
\end{align*}
\]

\[
\begin{align*}
r &= _______ \\
\end{align*}
\]

\[
\begin{align*}
r &= _______ \\
\end{align*}
\]

Description: __________

Center: _______

Radius: _______

Starts at _______

Direction: _______

c. \( r = a \cos \theta, \ 0 \leq \theta \leq 2\pi \) where \( a = 2, 3, -2 \). Answer the following for each given value of \( a \).

\[
\begin{align*}
r &= _______ \\
\end{align*}
\]

\[
\begin{align*}
r &= _______ \\
\end{align*}
\]

\[
\begin{align*}
r &= _______ \\
\end{align*}
\]

Description: __________

Center: _______

Radius: _______

Starts at _______

Direction: _______
Polar Exploration

d. Conclusions:
For \( r = a \sin \theta, \; 0 \leq \theta \leq 2\pi \), the graph is a _______ with center _______ and radius _______.

The graph is shifted _______ if \( a > 0 \) and _______ if \( a < 0 \).

For \( r = a \cos \theta, \; 0 \leq \theta \leq 2\pi \), the graph is a _______ with center _______ and radius _______.

The graph is shifted _______ if \( a > 0 \) and _______ if \( a < 0 \).

The graphs trace out _____________________.

Experiment #2:
a. \( r = a \sin (b\theta), \; 0 \leq \theta \leq 2\pi \) where \( a = 3 \) and \( b = 2 \).

\( r = \) ___________________

Description: _______ Length of petal (use Pythagorean Theorem)

Quadrant order: _________________ Trace to name four polar coordinates in terms of \( \pi \) for the rectangular coordinate \( (0, 0) \):

Direction: ______________________

Number of times traced: _____

b. \( r = a \sin (b\theta), \; 0 \leq \theta \leq 2\pi \) where \( a = 3 \) and \( b = 4 \);

\( r = \) ___________________

Description: _______ Length of petal (use Pythagorean Theorem)

Quadrant order: _________________ Trace to name four different polar coordinates than part (a) in terms of \( \pi \) for the rectangular coordinate \( (0, 0) \):

Direction: ______________________

Number of times traced: _____
Polar Exploration

c. \( r = a \sin (b\theta) \), \( 0 \leq \theta \leq 2\pi \) where \( a = 3 \) and \( b = 3 \); \( a = 3 \) and \( b = 5 \); \( a = -3 \) and \( b = 3 \); \( a = -3 \) and \( b = 2 \);
\( a = 3 \) and \( b = -3 \); and \( a = 3 \) and \( b = -2 \)

\[
\begin{align*}
\text{Description: } & \quad \text{Description: } \quad \text{Description: } \\
\text{Quadrant order: } & \quad \text{Quadrant order: } \quad \text{Quadrant order: } \\
\text{Direction: } & \quad \text{Direction: } \quad \text{Direction: } \\
\text{Number of times traced: } & \quad \text{Number of times traced: } \quad \text{Number of times traced: } \\
\text{Length of petal: } & \quad \text{Length of petal: } \quad \text{Length of petal: }
\end{align*}
\]

\[
\begin{align*}
\text{Description: } & \quad \text{Description: } \quad \text{Description: } \\
\text{Quadrant order: } & \quad \text{Quadrant order: } \quad \text{Quadrant order: } \\
\text{Direction: } & \quad \text{Direction: } \quad \text{Direction: } \\
\text{Number of times traced: } & \quad \text{Number of times traced: } \quad \text{Number of times traced: } \\
\text{Length of petal: } & \quad \text{Length of petal: } \quad \text{Length of petal: }
\end{align*}
\]
Polar Exploration

d. Answer the following

How do you determine the number of petals from the equation?

How can you determine the length of the petal from the equation?

How do you determine how many times the graph is traced?

How do you determine in which quadrant the graph will start?

What transformation occurs when $a$ is negative?

What transformation occurs when $b$ is negative?

e. $r = a \cos(b \theta)$, $0 \leq \theta \leq 2\pi$ where $a = 3$ and $b = 3$; $a = -3$ and $b = 3$; $a = 3$ and $b = -3$

$r = \_\_\_\_\_\_\_\_\_\_$

$r = \_\_\_\_\_\_\_\_\_$

$r = \_\_\_\_\_\_\_\_\_$

Description: \_\_\_\_\_\_\_\_\_\_

Quadrant order: \_\_\_\_\_\_\_\_\_\_

Direction: \_\_\_\_\_\_\_\_\_\_\_

Number of times traced: \_

Length of petal: \_

f. Do the conjectures concerning sine hold true for cosine as well? If there are differences, discuss them. Be sure to consider when $b$ is even as well.

\[ r = 5 \sin(5\theta) \]

\[ r = 4 \cos(2\theta) \]

\[ r = -6 \cos(6\theta) \]

g. Without a calculator, graph the following on $0 \leq \theta \leq 2\pi$
Polar Exploration

Experiment #3: The following equation graphs a limacon (lē-mə-ˈson) with a loop: \( r = a + b \sin(\theta), \ |a| < |b| \). You may have to change the window to see the entire graph; after you adjust the window, use zoom square.

a. Sketch each graph for \( 0 \leq \theta \leq 2\pi \)

\[ r = 2 + 4 \sin\theta \]

\[ r = 3 + 6 \sin\theta \]

\[ r = 3 + 5 \sin\theta \]

Starts at _______  
Starts at _______  
Starts at _______  

Direction: ______________  
Direction: ______________  
Direction: ______________

Number of times traced: _____  
Number of times traced: _____  
Number of times traced: _____

Least distance from pole: ____  
Least distance from pole: ____  
Least distance from pole: ____

Greatest distance from pole: ___  
Greatest distance from pole: ___  
Greatest distance from pole: ___

\[ r = 3 - 5 \sin\theta \]

\[ r = -3 + 5 \sin\theta \]

Starts at _______  
Starts at _______  

Direction: ______________  
Direction: ______________

Number of times traced: _____  
Number of times traced: _____

Least distance from pole: ____  
Least distance from pole: ____

Greatest distance from pole: ___  
Greatest distance from pole: ___

b. Answer the following:

How do you determine how far the loop extends from the pole from the equation?

How do you determine the greatest distance from the pole from the equation?

What happens if \( b \) is negative?

What happens if \( a \) is negative?
Polar Exploration

Experiment #4: The following equation graphs a limacon with no loop: \( r = a + b \sin(\theta), \ |a| > |b| \). There are two types of limacons with no loops, dimpled and convex. The first three are dimpled; the last two, convex.

a. Sketch each graph for \( 0 \leq \theta \leq 2\pi \)

\[ r = 4 + 3\sin \theta \]  
\[ r = 4 - 3\sin \theta \]  
\[ r = -4 + 3\sin \theta \]

Starts at _______  
Starts at _______  
Starts at _______

Direction: _______________  
Direction: _______________  
Direction: _______________

Number of times traced: _____  
Number of times traced: _____  
Number of times traced: _____

Least distance from pole: ___  
Least distance from pole: ___  
Least distance from pole: ___

Greatest distance from pole: ____  
Greatest distance from pole: ____  
Greatest distance from pole: ___

\[ r = 4 + 2\sin \theta \]  
\[ r = 3 - \sin \theta \]

Starts at _______  
Starts at _______

Direction: _______________  
Direction: _______________

Number of times traced: _____  
Number of times traced: _____

Least distance from pole: ___  
Least distance from pole: ___

Greatest distance from pole: ____  
Greatest distance from pole: ___

b. Answer the following:

Do your conjectures about limacons with a loop hold true for dimpled limacons? For convex limacon? If not, make new conjectures that apply to each type.

c. Graph the following without a calculator.

\[ r = 3 + 2\sin \theta \]  
\[ r = 2 - \sin \theta \]  
\[ r = -3 + 5\sin \theta \]

Starts at _______  
Starts at _______  
Starts at _______
Polar Exploration

Experiment #5: The following equation graphs a cardioid: \( r = a + b \sin(\theta), \ a = b \).

a. Sketch each graph for \( 0 \leq \theta \leq 2\pi \).
   
   \( r = 4 + 4\sin\theta \)  
   \( r = 5 + 5\sin\theta \)  
   \( r = -2 - 2\sin\theta \)

   Starts at _______  
   Starts at _______  
   Starts at _______

   Direction: _______________  
   Direction: _______________  
   Direction: _______________

   Number of times traced: _____  
   Number of times traced: _____  
   Number of times traced: ____

   At (0, 0) at \( \theta = _____ \)  
   At (0, 0) at \( \theta = _____ \)  
   At (0, 0) at \( \theta = _____ \)

   Distance from pole: ____  
   Distance from pole: ____  
   Distance from pole: ____

b. Answer the following:
   How do you determine the distance from the pole from the equation?

   What happens if \( a \) and \( b \) are negative?

   Which conjectures will hold true for limicons and cardioids?

c. Graph the following without a calculator.
   
   \( r = 3 + 3\sin\theta \)  
   \( r = -3 - 3\sin\theta \)
Polar Exploration

Experiment #6: The following equations graph lemniscates \(( r^2 = a^2 \sin 2\theta \) and \( r^2 = a^2 \cos 2\theta \)). These graphs look like rose curves with two petals. They are continuous on their domains. When you graph on the calculator, use \(0 \leq \theta \leq 2\pi\), but you may need to adjust your \(\theta\) step to produce an entire smooth graph.

a. \( r = 3\sqrt{\sin 2\theta} \)

\[\text{Starts at ________} \]
Maximum distance from the pole: _____
Domain: __________________________
Range: ____________________________

\( r = -3\sqrt{\sin 2\theta} \)

\[\text{Starts at ________} \]
Maximum distance from the pole: _____
Domain: __________________________
Range: ____________________________

b. Answer the following

How do you determine the length of the loop?

How many loops are in the lemniscate? Where does this number appear in the equation?

What happens is \(a\) is negative?

Experiment #7: The following equation graphs the Spiral of Archimedes

a. Sketch the graph of \( r = \theta \)

\[0 \leq \theta \leq 2\pi \quad 0 \leq \theta \leq 4\pi\]

b. Discuss how the domain affects the graph.
Polar Exploration

Experiment #8: Comparison between POLAR and FUNCTION mode.

a. For each of the following
   • Name the type of graph based on the equation.
   • Sketch the polar graph without using the calculator on \( 0 \leq \theta \leq 2\pi \)
   • Sketch the function graph.

\[
egin{align*}
  r &= 7 + 5 \sin \theta \\
  r &= 5 + 5 \sin \theta \\
  r &= 3 + 5 \sin \theta \\
  r &= 3 \cos 2\theta
\end{align*}
\]

Describe the relationship between the polar graphs and their corresponding function graphs.
**Polar Exploration**

VII. Identify the type of graph and the sketch the graph the following without using a calculator.

a. \( r = 4 \sin \theta \) 

b. \( r = 2 + 3 \sin \theta \) 

c. \( r = 2 \sin(2\theta) \) 

\[ \]

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Polar Exploration

Copy of special polar graphs from your textbook, page 735.

Special Polar Graphs

Several important types of graphs have equations that are simpler in polar form than in rectangular form. For example, the polar equation of a circle having a radius of $a$ and centered at the origin is simply $r = a$. Later in the text you will come to appreciate this benefit. For now, several other types of graphs that have simpler equations in polar form are shown below. (Conics are considered in Section 10.6.)

**Limaçons**

$r = a \pm b \cos \theta$

$r = a \pm b \sin \theta$

($a > 0, b > 0$)

$Limaçon with inner loop$

$Limaçon with outer loop$

Cardioid (heart-shaped)

Dimpled limaçon

Convex limaçon

**Rose Curves**

$n$ petals if $n$ is odd

$2n$ petals if $n$ is even ($n \geq 2$)

$r = a \cos n\theta$

Rose curve

$r = a \sin n\theta$

Rose curve

$r = \cos n\theta$

Rose curve

$r = \sin n\theta$

Rose curve

**Circles and Lemniscates**

$r = a \cos \theta$

Circle

$r = a \sin \theta$

Circle

$r^2 = a^2 \cos 2\theta$

Lemniscate

$r^2 = a^2 \sin 2\theta$

Lemniscate